Stochastic Processes and their Applications 4 (1976) 41–54

ON A MARKOV PROCESS GENERATED BY NON-DECREASING CONCAVE FUNCTIONS*

Joseph A. YAHAV
Department of Statistics, Columbia University, New York, N.Y. 10027, U.S.A.

Received 3 February 1975

A discrete-time Markov process on $[0, \infty)$ is considered. The process is generated by selecting at each time, in an independent and stationary way, a concave non-decreasing function. Sufficient conditions for the existence of a unique stationary limiting distribution are given.

Markov process	continuous state space
stationary distribution	fixed point
limiting distribution	concave functions

0. Introduction

In [2] I have considered a Markov process generated by non-decreasing functions on $[0, 1]$. To be more specific, let A be an arbitrary index set and $\{f_a : a \in A\}$ a collection of non-decreasing functions from $[0, 1]$ into $[0, 1]$. Let $\{\alpha_i : i = 1, 2 \ldots\}$ be a sequence of i.i.d. elements taking values in A. Let

$$X_n(y) = f_{\alpha_n}(f_{\alpha_{n-1}}(\ldots f_{\alpha_1}(y))) ;$$

then $X_n(y)$ is a Markov process for y in $[0, 1]$. In [2] I have given sufficient conditions for the existence of a unique limiting stationary distribution. Dubins and Freedman [1] consider Markov processes generated in the same way. In this paper I investigate the situation where I remove the restriction of compactness on the state space, i.e., I let $f_a(\cdot)$, $a \in A$, be a non-decreasing function from $[0, \infty)$ into $[0, \infty)$.

* Research supported in part by the National Science Foundation under Grant NSF-GP-33570x at Columbia University.

I give answers to questions on the behaviour of $X_n(y)$ when $f_a(\cdot)$ is assumed to be concave.

In Section 1, some lemmas regarding $X_n(y)$ and $\mathrm{E}\{f_{\alpha_1}(y)\}$ are proved. In Section 2, I classify the limiting behaviour of $X_n(y)$ according to the behaviour of $\mathrm{E}\{f_{\alpha_1}(y)\}$. In Section 3, some examples and possible applications are given.

1. Preliminaries and some lemmas

We start by formulating the assumptions on the structure of the process.

A1. A is an arbitrary index set. For every $a \in A$, $f_a(\cdot)$ is a non-decreasing concave function from $[0,\infty)$ into $[0,\infty)$.

A2. $(\Omega, \mathscr{A}, P)$ is a probability space. $\{\alpha_i, i = 1, 2 \ldots\}$ is a sequence of i.i.d. random elements taking values in A. The sets $\{w: f_{\alpha_1}(y) \leqslant x\}$ are in $\mathscr{A}$ for all non-negative reals x and y.

Let

$$X_n(y) = f_{\alpha_n}(f_{\alpha_{n-1}}(\ldots f_{\alpha_1}(y))) ,$$

$$g(y) = \mathrm{E}\{f_{\alpha_1}(y)\}, \qquad g^{(n)}(y) = g(g^{(n-1)}(y)), \quad n = 2,3, \ldots ,$$

$$H_y^{(n)}(x) = \mathrm{P}[X_n(y) \leqslant x] .$$

Lemma 1.1. *$g(y)$ is non-decreasing and concave. If* $\mathrm{P}[\{f_{\alpha_1}$ *is strictly concave*$\}] > 0$, *then $g(y)$ is strictly concave.*

Proof. For $0 \leqslant \gamma \leqslant 1$ and $y_1 < y_2$,

$$g(y_1) + (1-\gamma) g(y_2) = \mathrm{E}\{\gamma f_{\alpha_1}(y_1) + (1-\gamma) f_{\alpha_1}(y_2)\} . \qquad (1.1)$$

Since f_{α_1} is a.s concave, we have

$$\gamma f_{\alpha_1}(y) + (1-\gamma) f_{\alpha_1}(y_2) \leqslant f_{\alpha_1}(\gamma y_1 + (1-\gamma) y_2) \quad \text{a.s,} \qquad (1.2)$$

and hence

$$\gamma g(y_1) + (1-\gamma) g(y_2) \leqslant g(\gamma y_1 + (1-\gamma) y_2) .$$

If $\mathbf{P}[\{f_{\alpha_1}$ is strictly concave$\}] > 0$, we have strict inequality in (1.2) on a set with positive probability, which yields strict inequality in (1.3). The claim that $g(y)$ is non-decreasing follows from the assumption that $f_\alpha(y)$ is a.s. non-decreasing. □

Lemma 1.2. $\mathbf{P}[X_n(0) \leqslant x]$ *is non-increasing in n and hence converges.*

Proof. Let

$$\widetilde{X}_n(0) = f_{\alpha_1}(f_{\alpha_2} \dots (f_{\alpha_n}(0)));$$

then $\widetilde{X}_n(0)$ is a.s. non-decreasing. Since α_i are i.i.d., $\widetilde{X}_n(0)$ and $X_n(0)$ have the same distribution, and the lemma follows. □

Lemma 1.3. *If* $f_1, f_2, \dots, f_n$ *are non-decreasing concave functions, then* $f_1(f_2(\dots f_n(\cdot)))$ *is a non-decreasing concave function.*

Proof. We shall prove the result for $n = 2$; the conclusion for $n > 2$ follows by induction. Let $0 \leqslant \gamma \leqslant 1$, $y_1 < y_2$. Then since f_1 is non-decreasing and f_2 concave,

$$f_1(f_2(\gamma y_1 + (1-\gamma)y_2)) \geqslant f_1(\gamma f_2(y_1) + (1-\gamma) f_2(y_2)).$$

Since f_1 is concave, the concavity of $f_1(f_2(\cdot))$ follows. □

Lemma 1.4. *If* $g(0) > 0$ *and* $g(y_0) = y_0$, *then for every* $y \geqslant y_0$, $\mathbf{P}[X_n(y) \leqslant x]$ *converges say to* $F_y(x)$. $\mathrm{E}\{X_n(y)\}$ *converges, with*

$$\lim \mathrm{E}\{X_n(y)\} = \int_0^\infty \ldots \mathrm{d}F_y(x) \leqslant y_0 .$$

Proof. Let

$$\widetilde{X}_n(y) = f_{\alpha_1}(f_{\alpha_2}(\dots f_{\alpha_n}(y))),$$

and let B_n be the σ-field generated by $\alpha_1, \dots, \alpha_n$. Then

$$\mathrm{E}\{\widetilde{X}_{n+1}(y)/B_n\} = \mathrm{E}\{f_{\alpha_1}(f_{\alpha_2}(\dots f_{\alpha_{n+1}}(y)))/B_n\}.$$

Since by Lemma 1.3, $f_{\alpha_1}(f_{\alpha_2} \cdots f_{\alpha_n}(\cdot)))$ is a.s. concave, we get

$$\mathrm{E}\{\widetilde{X}_{n+1}(y)/B_n\} \leqslant f_{\alpha_1}(f_{\alpha_2} \cdots (f_{\alpha_n}(\mathrm{E}\{f_{\alpha_{n+1}}(y)\})))$$
$$\leqslant f_{\alpha_1}(f_{\alpha_2}(\ldots f_{\alpha_n}(g(y)))) \quad \text{a.s.} \tag{1.4}$$

Since $g(\cdot)$ is non-decreasing and concave, we have $g(y) \leqslant y$ for $y \geqslant y_0$. Since $f_{\alpha_1}(f_{\alpha_2}(\ldots f_{\alpha_n}(\cdot)))$ is a.s. non-decreasing, we get

$$\mathrm{E}\{\widetilde{X}_{n+1}(y)/B_n\} \leqslant f_{\alpha_1}(f_{\alpha_2}(\ldots f_{\alpha_n}(y))) = \widetilde{X}_n(y) \quad \text{a.s.}$$

We can then conclude that $\widetilde{X}_n(y)$ $(y \geqslant y_0)$ is a non-negative super-martingale with $0 \leqslant \mathrm{E}\{\widetilde{X}_n(y)\} \leqslant y$ for all n. Hence $\widetilde{X}_n(y)$ converges a.s. and L_1.

It follows from (1.4) that

$$\mathrm{E}\{\widetilde{X}_{n+1}(y)\} \leqslant g(g(\ldots g(y))) = g^{(n+1)}(y). \tag{1.5}$$

Since $g(\cdot)$ is a non-decreasing concave function and $g(0) > 0$,

$$\lim_n g^{(n)}(y) = y_0 ,$$

for every y and hence

$$\lim \mathrm{E}\{\widetilde{X}_n(y)\} \leqslant y_0 .$$

Since $X_n(y)$ and $\widetilde{X}_n(y)$ have the same distribution, the lemma follows. □

Lemma 1.5. *If* $g(0) > 0$, $g(y_0) = y_0$, *then for all* $\epsilon > 0$ *and* $n \geqslant n_1(\epsilon)$,

$$\mathrm{P}[X_n(0) \geqslant y_0 - \epsilon] > 0 . \tag{1.6}$$

Proof. Note first that

$$\mathrm{P}[f_{\alpha_i}(y) \geqslant g(y)] > 0$$

for all i and y. Now

$$\begin{aligned}
\mathrm{P}[X_n(0) \geq g^{(n)}(0)] &\geq \mathrm{P}[X_1(0) \geq g(0), X_2(0) \geq g^{(2)}(0), \ldots, X_n(0) \geq g^{(n)}(0)] \\
&= \mathrm{P}[f_{\alpha_1}(0) \geq g(0)] \\
&\quad \times \mathrm{P}[f_{\alpha_2}(X_1(0)) \geq g^{(2)}(0) \mid X_1(0) \geq g(0)] \ldots \\
&\quad \times \mathrm{P}[f_{\alpha_n}(X_{n-1}(0)) \geq g^{(n)}(0) \mid X_1(0) \geq g(0), \ldots, \\
&\qquad\qquad X_{n-1}(0) \geq g^{(n-1)}(0)] \\
&\geq \mathrm{P}[f_{\alpha_1}(0) \geq g(0)]\, \mathrm{P}[f_{\alpha_2}(g(0)) \geq g^{(2)}(0)] \ldots \\
&\quad \times \mathrm{P}[f_{\alpha_n}(g^{(n-1)}(0)) \geq g^{(n)}(0)] > 0 \,.
\end{aligned}$$

The second inequality follows from the assumption that $f_\alpha(\cdot)$ is a.s. non-decreasing. Now $g^{(n)}(0) \uparrow y_0$, and hence for every $\epsilon > 0$ there exists $n_1(\epsilon)$ such that $\mathrm{P}[X_{n_1}(0) \geq y_0 - \epsilon] > 0$; but from Lemma 1.2 we have that $\mathrm{P}[X_n(0) \geq x]$ is non-decreasing, and hence the lemma follows. □

Lemma 1.5. *If* $g(0) > 0$, $g(y_0) = y_0$ *and* $\mathrm{P}[f_{\alpha_1}(y_0) > y_0] > 0$, *then, for* $\delta > 0$ *sufficiently small,*

$$\mathrm{P}[X_n(0) \geq y_0 + \delta \text{ for some } n] = 1 \,. \tag{1.7}$$

Proof. Since $f_{\alpha_1}(\cdot)$ is a.s. continuous at y_0, (1.6) with $\epsilon > 0$ sufficiently small and $\mathrm{P}[f_{\alpha_1}(y_0) > y_0] > 0$ implies that there exist an n_2 such that for $\delta > 0$ sufficiently small $\mathrm{P}[X_{n_2}(0) \geq y_0 + \delta] > 0$. Since $\mathrm{P}[X_n(0) \geq x]$ is non-decreasing, we obtain for all $n \geq n_2$,

$$\mathrm{P}[X_n(0) \geq y_0 + \delta] \geq \mathrm{P}[X_{n_2}(0) \geq y_0 + \delta] > 0 \,. \tag{1.8}$$

Now

$$\begin{aligned}
&\mathrm{P}[X_n(0) < y_0 + \delta \text{ for all } n] \leq \\
&\quad \leq \mathrm{P}[X_{mn_2}(0) < y_0 + \delta \text{ for every } m = 1, 2, \ldots, k] \\
&\quad = \mathrm{P}[X_{n_2}(0) < y_0 + \delta]\, \mathrm{P}[X_{2n_1}(0) < y_0 + \delta \mid X_{n_2}(0) < y_0 + \delta] \ldots \\
&\quad\quad \times \mathrm{P}[X_{kn_2}(0) < y_0 + \delta \mid X_{jn_2}(0) < y_0 + \delta \text{ for } j = 1, 2, \ldots, k-1] \,.
\end{aligned}$$

Also for all j we have

$$\mathrm{P}[X_{jn_2}(0) < y_0 + \delta \mid x_{in_2}(0) < y_0 + \delta, i = 1, 2, ..., j-1] \leqslant$$
$$\leqslant \mathrm{P}[X_{n_2}(0) < y_0 + \delta] \,. \tag{1.9}$$

Using (1.9) we get

$$\mathrm{P}[X_n(0) < y_0 + \delta \text{ for all } n] \leqslant \{\mathrm{P}[X_{n_2}(0) < y_0 + \delta]\}^k \,. \tag{1.10}$$

Since (1.10) holds for all k, using the right-hand side of (1.8) yields (1.7). □

Let

$$\hat{f}_a(y) = r_a y + b_a \,.$$

For $a \in A$ and $y_0 \in (0, \infty)$ we can find r_a and b_a such that $\hat{f}_a(y) \geqslant f_a(y)$ and $\hat{f}_a(y_0) = f_a(y_0)$. Let $X_n^*(y)$ denote the process generated by $\{\hat{f}_a(\cdot): a \in A\}$ and $g^*(y) = \mathrm{E}\{\hat{f}_{\alpha_1}(y)\}$.

Lemma 1.7. *If $g(0) > 0$ and $g(y_0) = y_0$, then $g^*(0) > 0$, $g^*(y_0) = y_0$ and for $y \geqslant y_0$, $X_n^*(y)$ converges in law say to a limiting distribution $F_y^*(\cdot)$ with*

$$\lim_n \mathrm{E}\{X_n^*(y)\} = \int_0^\infty x \,\mathrm{d}F_y^*(x) \leqslant y_0 \,.$$

Proof. $\hat{f}_{\alpha_1}(0) \geqslant f_{\alpha_1}(0)$ a.s. implies $g^*(0) \geqslant g(0)$ and hence $g^*(0) > 0$; $\hat{f}_{\alpha_1}(y_0) = f_{\alpha_1}(y_0)$ a.s. implies $g^*(y_0) = g(y_0) = y_0$. Therefore, since $\hat{f}_{\alpha_1}(\cdot)$ is concave (in a wide sense), Lemma 1.4 holds and the above lemma follows. □

Lemma 1.8. *If $g(0) > 0$ and $g(y_0) = y_0$, then*

$$\prod_{i=1}^n r_{\alpha_i} \to 0 \quad a.s.$$

Proof. Since $g^*(0) > 0$, $g^*(y_0) = y_0$, and $\mathrm{E}\{r_{\alpha_i}\}\, y_0 + \mathrm{E}\{b_{\alpha_i}\} = y_0$, we get $\mathrm{E}\{r_{\alpha_i}\} < 1$ and hence $\mathrm{E}\,\{\log r_{\alpha_i}\} < 0$. By the strong law of large numbers,

$$\sum_{i=1}^{n} \log r_{\alpha_i} \to -\infty \quad \text{a.s.,}$$

and hence the lemma follows. □

Lemma 1.9. *If $g(0) > 0$ and $g(y_0) = y_0$, then for all y, $\lim_n \mathrm{P}[X_n^*(y) \leqslant x]$ exists and is independent of y.*

Proof. We have

$$X_n^*(y) = y \prod_{i=1}^{n} r_{\alpha_i} + b_{\alpha_n} + \sum_{j=1}^{n-1} b_{\alpha_j} \prod_{i=j+1}^{n} r_{\alpha_i} \,. \tag{1.15}$$

Using Lemmas 1.7 and 1.8 completes the proof. □

Lemma 1.10. *If $g(0) > 0$ and $g(y_0) = y_0$, then $F^*(x) = \lim_n \mathrm{P}[X_n^*(y) \leqslant x]$ is the unique stationary distribution for the process generated by $\{\hat{f}_{\alpha_1}(y)\}$.*

Proof. To prove the stationarity of $F^*(x)$ we have to show that

$$F^*(x) = \int_0^\infty \mathrm{P}[\hat{f}_{\alpha_1}(y) \leqslant x] \; \mathrm{d}F^*(y) \,. \tag{1.16}$$

Let

$$H_y^{*(n)}(x) = \mathrm{P}[X_n^*(y) \leqslant x] \,;$$

we have then

$$H_0^{*(n)}(x) = \int_0^\infty H_y^{(1)}(x) \; \mathrm{d}H_0^{*(n-1)}(y) \,. \tag{1.17}$$

Since $\hat{f}_{\alpha_1}(y)$ is a.s. continuous in y, so is $H_y^{(1)}(x)$, and this latter being continuous and uniformly bounded entails the use of the Helly Bray lemma, which completes the proof of stationarity. The uniqueness follows from the fact that if $G(x)$ is a stationary distribution for the process, then

$$\int_0^\infty H_y^{*(n)}(x) \; \mathrm{d}G(y) = G(x) \quad \text{for all } n \,.$$

But $H_y^{*(n)}(x) \to F^*(x)$ for all y, hence in the limit we get $G(x) = F^*(x)$. □

Lemma 1.11. *If $g(0) > 0$ and $g(y_0) = y_0$, then for every $\delta > 0$ and all n sufficiently large we have*

$$P[X_n^*(y) \leqslant y_0 + \delta] > 0, \qquad \lim_n P[X_n^*(y) \leqslant y_0 + \delta] > 0, \quad (1.18)$$

$$P[X_n^*(y) \leqslant y_0 + \delta \text{ i.0}] = 1 . \quad (1.19)$$

Proof. To prove (1.18), recall from Lemma 1.7 that

$$\lim_n E[X_n^*(y)] = \int_0^\infty x \, dF^*(x) \leqslant y_0 .$$

To prove (1.19) note that the event $\{X_n^*(y) \leqslant y_0 + \delta \text{ i.0}\}$ is an invariant tail event. Since $X_n^*(y)$ is a Markov process with a unique limiting stationary distribution, (1.18) implies (1.19). □

Lemma 1.12. *If $g(0) > 0$ and $g(y_0) = y_0$, then for every $\delta > 0$ and for all sufficiently large n,*

$$P[X_n(y) \leqslant y_0 + \delta] > 0, \qquad \lim_n P[X_n(y) \leqslant y_0 + \delta] > 0 \quad (1.20)$$

$$P[X_n(y) \leqslant y_0 + \delta \text{ i.0}] = 1 . \quad (1.21)$$

Proof. $X_n(y) \leqslant X_n^*(y)$ a.s.; using Lemma 1.11 the required results follow. □

2. Theorems

We shall start first by investigating the case:

C1. $g(0) > 0$, $g(y_0) = y_0$, $P[f_{\alpha_1}(y_0) > y_0] > 0$, $P[f_{\alpha_1}(\cdot)$ continuous at $0] = 1$.

Theorem 2.1. *Under* A1, A2, C1, *the process $\{X_n(y): n = 1, 2, \ldots\}$ has a unique limiting stationary distribution.*

Proof. Let

$$H_0^{(n)}(x) = \mathrm{P}[X_n(0) \leqslant x], \qquad H_y^{(n)}(x) = \mathrm{P}[X_n(y) \leqslant x] ,$$

$$F_0(x) = \lim_n H_0^{(n)}(x), \qquad F_{y_0+\delta}(x) = \lim_n \mathrm{P}[X_n(y_0+\delta) \leqslant x] .$$

Define

$$\tau_1 = \inf\{n\colon X_n(0) \geqslant y_0+\delta\} ,$$

where δ is chosen to satisfy (1.7). Then $\mathrm{P}[\tau_1 < \infty] = 1$. Now we can write

$$\mathrm{P}[X_n(0) \leqslant x] = \sum_{i=1}^{n} \mathrm{P}[X_n(0) \leqslant x,\ \tau_1 = i] + \mathrm{P}[X_n(0) \leqslant x, \tau > n]$$

$$= \sum_{i=1}^{n} \int_{y_0+\delta}^{\infty} H_y^{(n-i)}(x)\, \mathrm{d}Q_1(y \mid \tau = i) + \mathrm{P}[X_n(0) \leqslant x,\ \tau > n] ,$$

where

$$Q_1(y \mid \tau_1 = i) = \mathrm{P}[X_{\tau_1}(0) \leqslant y,\ \tau_1 = i] .$$

Since $H_y^{(n-1)}(x) \leqslant H_{y_0+\delta}^{(n-1)}(x)$ for all y, $y \geqslant y_0+\delta$, we can write

$$\mathrm{P}[X_n(0) \leqslant x] \leqslant \sum_{i=1}^{n} H_{y_0+\delta}^{(n-1)}(x)\, \mathrm{P}[\tau_1 = i] + \mathrm{P}[X_n(0) \leqslant x, \tau_1 > n] . \tag{2.1}$$

By Lemmas 1.2 and 1.4 the limits of both sides of (2.1) exist and hence taking the limits we get

$$F_0(x) \leqslant F_{y_0+\delta}(x) . \tag{2.2}$$

But $H_0^{(n)}(x) \geqslant H_{y_0+\delta}^{(n)}(x)$ for all n and hence

$$F_0(x) = F_{y_0+\delta}(x) . \tag{2.3}$$

Now for any y, $0 \leqslant y \leqslant y_0 + \delta$, we have

$$H_0^{(n)}(x) \geqslant H_y^{(n)}(x) \geqslant H_{y_0+\delta}^{(n)} \geqslant H_{y_0+\delta}^{(n)}(x) . \tag{2.4}$$

Taking limits on both sides of (2.4) and using (2.3) we get for all $0 \leqslant y \leqslant y_0 + \delta$,

$$\lim_n \mathrm{P}[X_n(y) \leqslant x] = F_0(x) . \tag{2.5}$$

Let $y_1 > y_0 + \delta$, and define

$$\tau_2 = \inf\{n : x_n(y_1) \leqslant y_0 + \delta\} .$$

By Lemma 1.12, $\mathrm{P}[\tau_2 < \infty] = 1$. We can then write

$$\mathrm{P}[X_n(y_1) \leqslant x] = \sum_{i=1}^{n} \mathrm{P}[X_n(y_1) \leqslant x,\ \tau_2 = i] + \mathrm{P}[X_n(y_1) \leqslant x,\ \tau_2 > n]$$

$$= \sum_{i=1}^{n} \int_0^{y_0+\delta} H_y^{(n-1)}(x)\ \mathrm{d}Q_2(y \mid \tau_2 = i) + \mathrm{P}[X_n(y_1) \leqslant x, \tau_2 > n]$$

where

$$Q_2(y / \tau_2 = 1) = \mathrm{P}[X_n(y_1) \leqslant y,\ \tau_2 = i] .$$

Since, for $y \leqslant y_0 + \delta$, $H_y^{(n-1)}(x) \geqslant H_{y_0+\delta}^{(n-1)}(x)$, we have

$$\mathrm{P}[X_n(y_1) \leqslant x] \geqslant \sum_{i=1}^{n} H_{y_0+\delta}^{(n-1)}(x)\, \mathrm{P}[\tau_2 = i] + \mathrm{P}[X_n(y_1) \leqslant x, \tau_2 > n] . \tag{2.6}$$

By Lemma 1.4 the limits of both sides of (2.6) exist, and hence taking the limits we get

$$F_{y_1}(x) \geqslant F_{y_0+\delta}(x) . \tag{2.7}$$

Recalling that for all n, $H_{y_1}^{(n)}(x) \leqslant H_{y_0+\delta}^{(n)}(x)$, we get

$$F_{y_1}(x) = F_{y_0+\delta}(x) . \tag{2.8}$$

Since y_1 was arbitrary, using (2.3) we get for every $y, y \geqslant y_0 + \delta$,

$$F_y(x) = F_0(x) , \tag{2.9}$$

which yields together with (2.5) that for every y, $0 \leqslant y < \infty$,

$$\lim_n \mathrm{P}[X_n(y) \leqslant x] = F_0(x) . \tag{2.10}$$

To prove the stationarity of $F_0(x)$ recall that

$$H_0^{(n+1)}(x) = \int_0^\infty H_y^{(1)}(x)\, \mathrm{d}H_0^{(n)}(y)\,. \tag{2.11}$$

Since $f_\alpha(\cdot)$ is a.s. continuous, we get $H_y^{(1)}(x)$ to be continuous and uniformly bounded, and hence, by the Helly–Bray lemma we get

$$F_0(x) = \int_0^\infty H_y^{(1)}(x)\, \mathrm{d}F_0(y)\,. \tag{2.12}$$

To prove the uniqueness, let $G(x)$ be a stationary distribution; then, for every n,

$$G(x) = \int_0^\infty H_y^{(n)}(x)\, \mathrm{d}G(y)\,. \tag{2.13}$$

Hence, taking the limit in (2.13), we obtain

$$G(x) = \int_0^\infty F_0(x)\, \mathrm{d}G(y) = F_0(x). \square \tag{2.14}$$

Next we consider the case:

C2. $g(0) > 0, g(y_0) = y_0, \mathrm{P}[f_{\alpha_1}(y_0) = y_0] = 1$

Theorem 2.2. *Under* A1, A2, C2, $X_n(y) \to y_0$ *a.s. and* L_1.

Proof. Note first that for $y, y > y_0$, $\mathrm{P}[f_{\alpha_1}(y_0) = y_0] = 1$ implies that $\mathrm{P}[f_{\alpha_1}(y) \geqslant y_0] = 1$, and hence $\mathrm{P}[X_n(y) \geqslant y_0] = 1$ for all n. Recall from the proof of Lemma 1.4 that $\{\widetilde{X}_n(y)\}$ is distributed like $\{X_n(y)\}$, and $\widetilde{X}_n(y)$ convergences a.s. and in the first mean to a r.v. X with $\mathrm{E}\{X\} \leqslant y_0$. Hence for $y, y > y_0$, $\{\widetilde{X}_n(y) - y_0\}$ is a sequence of non-negative r.v. which converges a.s. and in the first mean to a r.v. with non-positive expectation. This implies $\widetilde{X}_n(y) - y_0 \to 0$ a.s. and L_1. For $y, y < y_0$, $\mathrm{P}[f_\alpha(y_0) = y_0] = 1$ with $g(0) > 0$ implies

$$\mathrm{P}(f_{\alpha_1}(y) \geqslant y) = 1, \qquad \mathrm{P}[f_{\alpha_1}(y) > y] > 0$$

and

$$\mathrm{P}[X_n(y) \leqslant y_0] = 1 \quad \text{for all } n\,.$$

Hence $\{X_n(y)\}$ is a sequence of non-negative r.v. which is monotone non-decreasing and uniformly bounded a.s.. Hence $X_n(y)$ converges a.s. and in the first mean to a r.v. $X(y)$. By Lemma 1.5 for n sufficiently large and all $\epsilon > 0$, $\mathrm{P}[X_n(0) \geqslant y_0 - \epsilon] > 0$. By an argument similar to the one used (from (1.8) to (1.10)) in the proof of Lemma 1.6 we get that for all $\epsilon > 0$,

$$\mathrm{P}[X_n(0) \geqslant y_0 - \epsilon \text{ for some } n] = 1$$

and so

$$\mathrm{P}[\lim_n X_n(0) \geqslant y_0 - \epsilon] = 1\ .$$

This implies that $\lim_n X_n(0) = y_0$ a.s. Since $X_n(0) \leqslant X_n(y) \leqslant y_0$ a.s., the proof is complete. □

Now the third case:

C3. $g(0) = 0$, $g(x) < x$ for all x, $x > 0$.

Theorem 2.3. *Under* A1, A2, C3, $X_n(y) \to 0$ *a.s. and in* L_1.

Proof. Here $y_0 = 0$ and the proof is similar to that of Theorem 2.2 for the case $y > y_0$. For all y, $X_n(y)$ is a non-negative supermartingale with $\lim_n \mathrm{E}\{X_n(y)\} = 0$. □

Another case where we are able to state a positive result is:

C4. $g(0) = 0$, $g(y_0) = y_0$ and there exists an $\epsilon > 0$ such that $\mathrm{P}[f_{\alpha_1}(\epsilon) \geqslant \epsilon] = 1$ and $g(\epsilon) > \epsilon$.

Theorem 2.4. *Under* A1, A2, C4, $X_n(y)$ *with* $y \geqslant \epsilon$ *converges in law to a unique limiting stationary distribution.*

Proof. C4 can fall under C1 or C2 with ϵ replacing 0.

3. Remarks and examples

In C1 (Theorem 2.1) we have the unpleasant assumption that

$$P[f_{\alpha_1}(\cdot) \text{ continuous at } 0] = 1 .$$

This was used only in the proof of the stationarity of the limiting distribution. My conjecture is that the theorem holds in that case without this assumption. There are two possible situations for which we do not have a positive answer: the first when $g(x) \geqslant x$ for all x, and the second when $g(0) = 0$ and there is no ϵ such that $P[f_{\alpha_1}(\epsilon) \geqslant \epsilon] = 1$ and $g(\epsilon) > \epsilon$. We will give here some examples to show what the difficulties involved are.

Example 3.1. Let $\{f_a : a \in A\}$ be of the form

$$f_a(x) = a_1 x + a_2 \log(1 + x) , \qquad a_1 > 0, \quad a_2 > 0 ,$$

and let $\alpha_n = (\alpha_{n1}, \alpha_{n2})$ be distributed so that $E\{\alpha_{n1}\} = 1$ and $E\{\lg(\alpha_{n1} + \alpha_{n2})\} < 0$. Then

$$g(x) = E\{\alpha_{11} x + \alpha_{12} \log(1 + x)\} = x + \log(1 + x)\, E\{\alpha_{12}\}$$

and hence $g(0) = 0$ and $g(x) > x$ for all $x > 0$. This may make us believe that $X_n(y) \to \infty$ for $y > 0$, but such is not the case for let $r_n = \alpha_{n1} + \alpha_{n2}$, and we have $f_{\alpha_n}(x) \leqslant r_n x$ a.s. Hence

$$X_n(y) \leqslant y \prod_{i=1}^{n} r_i \quad \text{a.s.},$$

but since $E\{\log r_i\} < 0$ we have $\Sigma \log r_i \to -\infty$ a.s. Therefore $\Pi_{i=1}^n r_i \to 0$ a.s., which yields $X_n(y) \to 0$ a.s.

Example 3.2. The same structure as in Example 3.1 but $E\{\log \alpha_{n1}\} > 0$. Here $f_{\alpha_n}(x) \geqslant \alpha_{n1} x$ a.s. and $X_n(y) \geqslant y \Pi_{i=1}^n \alpha_{11}$, but $\Sigma_{i=1}^n \log \alpha_{11} \to \infty$ a.s. and hence $X_n(y) \to \infty$ a.s.

Example 3.3. Let $f_a(x) = a_1 x + a_2$, $0 \leqslant a_1$, $0 \leqslant a_2$, and let $\alpha_n = (\alpha_{n1}, \alpha_{n2})$ distributed so that $E\{\alpha_{n1}\} = 1$ and $E\{\alpha_{n2}\} > 0$. We have $g(0) > 0$, $g(x) > x$ for all x, and

$$X_n(y) = Y \prod_{i=1}^{n} \alpha_{i1} + \alpha_{n2} + \sum_{j=1}^{n-1} \alpha_{j2} \prod_{i=j+1}^{n} \alpha_{i1} .$$

Since $E\{\alpha_{n1}\} = 1$ implies $E\{\log \alpha_{n1}\} < 0$, we get $\Pi_{i=1}^{n} \alpha_{i1} \to 0$ a.s. and $X_n(y)$ converges by law to a unique stationary limiting distribution.

A possible application is to an economic stochastic growth model. Consider a person starting at time 0 with capital k_0 and using a consumption policy determined by a function $C(\cdot)$ so that he is consuming $C(k_0)$ at time 0. His investment is then $k_0 - C(k_0)$. If he faces a random production function $f_\alpha(\cdot)$, then his capital at time 1 would be $k_1 = f_{\alpha_1}(k_0 - C(k_0))$, and at time n, $k_n = f_{\alpha_n}(k_{n-1} - C(k_{n-1}))$. If we assume $\{\alpha_i\}$ to be i.i.d. and $f_\alpha(\varphi(\cdot))$ to be a.s. non-decreasing and concave, where $\phi(k) = k - C(k)$, we are in the model discussed above.

References

[1] L.E. Dubins and D.A. Freedman, Invariant probabilities for certain Markov processes, Ann. Math. Statist. 37 (1966) 837–847.

[2] J.A. Yahav, On a fixed point theorem and its stochastic equivalent, J. Appl. Probab., to appear.